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Another Isomorphism Theorem on Anti-ordered Semigroups[1]

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Abstract

In this paper we give another isomorphism theorem on anti-ordered semigroups.

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1 Introduction and preliminaries

Our setting is the Bishop's constructive mathematics ([1], [2], [7]). Let $(S, =, \neq, \cdot)$ be a semigroup with apartness in the sense of paper [3], where '$\neq$' is a binary relation on X which satisfies the following properties

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$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq z \Longrightarrow x \neq y \ \vee\ y \neq z,$$
$$x \neq y \ \wedge\ y = z \Longrightarrow x \neq z$$

and

$$(\forall a, b, x \in S)((ax \neq bx \Longrightarrow a \neq b) \ \wedge\ (xa \neq xb \Longrightarrow a \neq b)).$$

The apartness is *tight* if $\neg(x \neq y) \Longrightarrow x = y$ holds. Let Y be a subset of S and $x \in S$. The subset Y of S is *strongly extensional* in S if and only if $y \in Y \Longrightarrow y \neq x \vee x \in Y$ holds ([2],[8]). If $x \in S$, we defined ([3]) $x \bowtie Y \Longleftrightarrow (\forall y \in Y)(y \neq x)$. Let $f : (S, =, \neq, \cdot) \longrightarrow (T, =, \neq, \cdot)$ be a mapping. We say that it is:
(a) f is *strongly extensional* if and only if $(\forall a, b \in S)(f(a) \neq f(b) \Longrightarrow a \neq b)$;
(b) f is an *embedding* if and only if $(\forall a, b \in S)(a \neq b \Longrightarrow f(a) \neq f(b))$.
Let $\alpha \subseteq S \times T$ and $\beta \subseteq T \times Z$ be relations. The *filled product* ([3]) of relations α and β is the relation

$$\beta * \alpha = \{(a, c) \in S \times Z : (\forall b \in T)((a, b) \in \alpha \ \vee\ (b, c) \in \beta\}.$$

A relation $q \subseteq S \times S$ is an *anticongruence relation* on S if and only if holds:

$$q \subseteq \neq,\ q \subseteq q^{-1},\ q \subseteq q * q,$$
$$(\forall a, b, x \in S)(((ax, bx) \in q \Longrightarrow (a, b) \in q) \ \wedge\ ((xa, xb) \in q \Longrightarrow (a, b) \in q)).$$

If q is an anticongruence on semigroup $(S, =, \neq, \cdot)$, we can construct factor-semigroup $(S/q, =_1, \neq_1, \cdot_1)$ with

$$aq =_1 bq \Longleftrightarrow (a, b) \bowtie q,\ aq \neq_1 bq \Longleftrightarrow (a, b) \in q,\ (aq) \cdot_1 (bq) = (ab)q.$$

A relation α on S is *antiorder* ([4]) on S if and only if

$$\alpha \subseteq \neq,\ \alpha \subseteq \alpha * \alpha,\ \neq \subseteq \alpha \cup \alpha^{-1},$$
$$(\forall a, b, x \in S)(((ax, bx) \in \alpha \Longrightarrow (a, b) \in \alpha) \ \wedge\ ((xa, xb) \in \alpha \Longrightarrow (a, b) \in \alpha)).$$

Let $f : (S, =, \neq, \cdot, \alpha) \longrightarrow (T, =, \neq, \cdot, \beta)$ be a strongly extensional homomorphism of ordered semigroups under antiorders. f is called *isotone* if $(\forall x, y \in S)((x, y) \in \alpha \Longrightarrow (f(x), f(y)) \in \beta)$; f is called *reverse isotone* if and only if $(\forall x, y \in S)((f(x), f(y)) \in \beta \Longrightarrow (x, y) \in \alpha)$. The strongly extensional mapping f is called an *isomorphism* if it is injective and embedding, onto, isotone and reverse isotone. S and T called *isomorphic*, in symbol $S \cong T$, if exists an isomorphism between them.
As in [4] a relation $\tau \subseteq S \times S$ is a *quasi-antiorder* on S if and only if

$$\tau \subseteq \alpha(\subseteq \neq),\ \tau \subseteq \tau * \tau$$
$$(\forall a, b, x \in S)(((ax, bx) \in \alpha \Longrightarrow (a, b) \in \alpha) \ \wedge\ ((xa, xb) \in \alpha \Longrightarrow (a, b) \in \alpha)).$$

For more information on these relations readers can see in the papers [4]-[7].

The first proposition gives some information about quasi-antiorders:

Lemma 1.1 *([4]) Let $(S, =, \neq, \cdot)$ be an anti-ordered semigroup and τ is a quasi-antiorder on S. Then, the relation $q = \tau \cup \tau^{-1}$ is an anticongruence on S, and $S/q = \{aq : a \in S\}$ with the anti-order $(aq, bq) \in \theta \Longleftrightarrow (a, b) \in \tau$ $(a, b \in S)$ is an anti-ordered semigroup and $\pi : S \longrightarrow S/q$, defined by $\pi(a) = aq$, is an reverse isotone strongly extensional homomorphism from S onto S/q.*

Lemma 1.2 *([6], [7]) Let $(S, =, \neq, \cdot, \alpha)$ and $(T, =, \neq, \cdot, \beta)$ be anti-ordered semigroups and $\varphi : S \longrightarrow T$ an reverse isotone strongly extensional homomorphism. Then,*

$$\varphi^{-1}(\beta) = \{(a, b) \in X \times X : (\varphi(a), \varphi(b)) \in \beta\}$$

is a quasi-antiorder on S with $\varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1} = Coker\varphi$, and if the apartness in T is tight, then $S/Coker \cong Im\varphi$ as anti-ordered sets.

Let $(S, =, \neq, \cdot, \alpha)$ be an anti-ordered semigroup. A quasi-antiorder σ on S is called a *quotient quasi-antiorder* (abbreviated to Q-quasi-antiorder) on S ([7]) if holds

$$\sigma \subseteq \alpha \subseteq q^C \circ \sigma \circ q^C.$$

If q is an anticongruence on anti-ordered semigroup $(S, =, \neq, \cdot, \alpha)$, then the semigroup S/q is not an anti-ordered semigroup, in general case. By result in [5], the relation $\pi \circ \alpha \circ \pi^{-1}$ is an antiorder on S/q if and only if the relation $\tau = q^C \circ \alpha \circ q^C$ is a quasi-antiorder on S such that $q = \tau \cup \tau^{-1}$. In papers [4], [6] and [7] the third author described some isomorphisms between ordered sets under antiorders. This paper is a continuation of [7]. Here, we give another isomorphism theorem on anti-ordered semigroups.

2 The Results

Let $(S, =, \neq, \cdot, \alpha)$ be an anti-ordered semigroup, τ a quasi-antiorder on S under α and $(T, =, \neq, \cdot)$ a subsemigroup of S. Let $T_s = \cup\{(\pi^{-1} \circ \pi)(x) : x \in T\}$. It is clear that $T \subseteq T_s$. It is clear that the following equivalence holds: $a \in T_s \Longleftrightarrow (\exists x \in T)((x, a) \bowtie q)$. Our first result is:

Theorem 2.1 *: Let $(S, =, \neq, \cdot, \alpha)$ be an anti-ordered semigroup such that $\theta = \pi \circ \alpha \circ \pi^{-1}$ is an anti-order on S/q where $q = \pi^{-1}(\theta) \cup (\pi^{-1}(\theta))^{-1}$ is an anticongruence on S. If T is a subsemigroup of S, then T_s is a subsemigroup of S and $\tau = \pi^{-1}(\theta)$ is a Q-quasi-antiorder relation on $(T_s, =, \neq, \alpha)$.*

Proof: Since T is a subsemigroup of S, then $\pi(T) = \{aq : a \in T\}$ is a subsemigroup of S/q. Hence $T_s = \pi^{-1}(\pi(T)) = \{\pi^{-1}(xq)) : x \in T\}$ is a subsemigroup of S.
Let $\eta = \pi|T_s$ be a restriction of π to T_s. Then η is a strongly extensional reverse isotone homomorphism of $(T_s, =, \neq, \cdot, \alpha)$ to $(S/q, =_1, \neq_1, \cdot_1, \theta)$. Since, for any $a, b \in T_s$, $(a, b) \in \eta^{-1}(\theta)$ is equivalent to $(\pi(a), \pi(b)) \in \theta$, i.e. to $(a, b) \in \tau$, and

$$\begin{aligned} &a(\eta^{-1}(\theta) \cup (\eta^{-1}(\theta))^{-1}) \\ &= \{x \in T_s : (a, x) \in \eta^{-1}(\theta) \ \vee \ (x, a) \in \eta^{-1}(\theta)\} \\ &= \{x \in T_s : (\pi(a), \pi(x)) \in \theta \ \vee \ (\pi(x), \pi(a)) \in \theta\} \\ &= \{x \in T_s : (a, x) \in \tau \ \vee \ (x, a) \in \tau\} \\ &= \{x \in S : x \in T_s \wedge ((a, x) \in \tau \ \vee \ (x, a) \in \tau)\} = aq \end{aligned}$$

we may regard $\tau = \eta^{-1}(\theta)$ as a quasi-antiorder on $(T_s, =, \neq, \cdot, \alpha)$. Indeed, for any a, b of T_s there exist elements x, y of T such that $a \in (\pi^{-1} \circ \pi)(x)$ and $b \in (\pi^{-1} \circ \pi)(y)$. Hence, out of $(a, b) \in \alpha$, we conclude that there exist elements x, y of T such that $((\pi^{-1} \circ \pi)(x), (\pi^{-1} \circ \pi)(y)) \in \alpha$ and $(\pi(x), \pi(y)) \in \pi \circ \alpha \circ \pi^{-1} = \theta$. Thus, $(x, y) \in \tau = \pi^{-1}(\theta)$. So, $(a, b) \in q^C \circ \pi^{-1}(\theta) \circ q^C$. Therefore, $\pi^{-1}(\theta)$ is a Q-quasi-antiorder on T_s. q.e.d.

The main result of this paper is the following theorem:

Theorem 2.2 *: Suppose that hypothesis' as in the Theorem 1. Then,* $\alpha_t = \alpha \cap (T \times T)$ *is an antiorder and* $\tau_t = \tau \cap (T \times T)$ *is a quasi-antiorder on* $(T, =, \neq, \cdot, \alpha_t)$ *such that* $\theta_t = \pi \circ \alpha_t \circ \pi^{-1}$ *is induced anti-order and, if the apartness on* S/q *is tight, the following isomorphism* $(T/q_t, =_1, \neq_1, \cdot_1, \theta_t) \cong (T_s/q, =_1, \neq_1, \cdot_1, \theta)$ *holds as anti-ordered semigroups, where* $q_t = \tau_t \cup (\tau_t)^{-1}$ *is corresponding anticongruence on* T.

Proof: By Theorem 2.1 and Lemma 1.1, $(T_s/q, =_1, \neq_1, \cdot_1, \theta)$ is an anti-ordered semigroup. Define $\pi_t : T \longrightarrow T_s/q$ by $\pi_t(a) = aq$ for any $a \in T$. Thus, π_t is a strongly extensional and surjective mapping. Indeed, if xq is an arbitrary element of T_s/q, then there exists an element x' of T such that $(x, x') \in q^C$ and $\pi_t(x') =_1 xq$. Further on, if xq and yq be elements of T_s/q with $xq \neq_1 yq$, then there exist elements $x', y' \in T$ such that $\pi_t(x') = xq$, $\pi_t(y') = yq$. Out of $(x, y) \in q \Longrightarrow (x, x') \in q \vee (x', y') \in q \subseteq \neq \vee (y', y) \in q$ we conclude $x' \neq y'$. For x, y of T, we have

$$\pi_t(xy) = (xy)q = (xq) \cdot_1 (yq) = \pi_t(x) \cdot_1 \pi_t(y); \text{ and}$$

$$(\pi_t(x), \pi_t(y)) \in \theta \Longleftrightarrow (xq, yq) \in \theta \Longleftrightarrow (x, y) \in \tau.$$

It is easy to check that α_t is an anti-order on T and $\theta_t = \pi \circ \alpha_t \circ \pi^{-1}$ on T/q_t respective. Thus, π_t is a strongly extensional isotone and reverse isotone epimorphism of $(T, =, \neq, \cdot, \alpha_t)$ onto $(T_s/q, =_1, \neq_1, \cdot_1, \theta)$. Since

$$\begin{aligned} &(\pi_t)^{-1}(\theta) = \{(x, y) \in T \times T : (\pi_t(x), \pi_t(y)) \in \theta\} \\ &= \{(x, y) \in T \times T : (qx, qy) \in \theta\} \end{aligned}$$

$$= \{(x, y) \in T \times T : (x, y) \in \tau\} = \tau \cap (T \times T) = \tau_t,$$

by Lemma 2 we conclude that τ_t is a quasi-antiorder on $(T, =, \neq, \cdot, \alpha_t)$. Let $x, y \in T_s$ ne arbitray elements of T_s such that $(xq, yq) \in \theta$. Then there exist elements x' and y' of T with $(x, x') \bowtie q$, $(y, y') \bowtie q$ and $(x, y) \in \tau \subseteq \alpha$. So, we have $\pi_t(x') = xq$, $\pi_t(y') = yq$ and $(x', y') \in \alpha$ Therefore, $(x', y') \in \alpha_t$. If the apartness $\neq_1$ on S/q is tight, $(T/q_t, =_1, \neq_1, \cdot_1, \theta_t) \cong (T_s/q, =_1, \neq_1, \cdot_1, \theta)$ as anti-ordered semigroups by Lemma 1.2. q.e.d.

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